also shown that interface location and temperature distribution depend on three parameters; $\beta = L/C_{p_s}(T_F - T_W)$, $T_i^* = (K_L/K_s)\ (T_i - T_F)/(T_F - T_W)$ and $\alpha = \alpha_s/\alpha_L$. The dimensionless temperatures in the solid and liquid are; $T_L^* = (K_L/K_s)\ (T_L - T_F)/(T_F - T_W)$ and $T_s^* = (T_s - T_F)/(T_F - T_W)$.

Table 1 gives a comparison between experimental and theoretical values of x_o^*, which is the interface location at $x^* = y^*$ for three different values of β and T_i^*. In each case the experimental value for x_o^* was calculated at different intervals of time. Results indicate that for a given β and T_i^*, x_o^* is nearly constant which confirms the similarity nature of the problem as predicted by theoretical consideration.

Figure 4 gives the liquid temperature transients at the diagonal $x^* = y^*$ for various locations. We observe that the profiles do not coalesce into a single curve as predicted by the theoretical solution. This is a direct consequence of the temperature inversion due to density variations of the liquid.

The one-dimensional behaviour of the system far away from the corner is illustrated in Fig. 5. Temperature measurements at $x = 10{\cdot}4$ in. from the corner are compared with Neumann's one-dimensional solution. Excellent agreement is observed in the solid region.

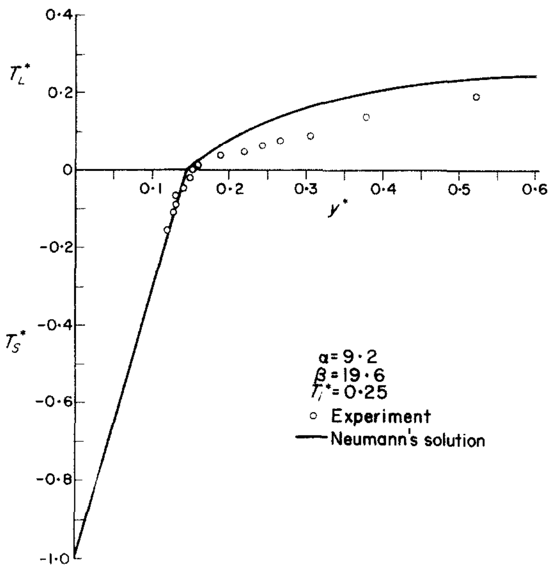

FIG. 5. Comparison between temperature measurements in the one-dimensional region ($x = 10{\cdot}4$ in.) and Neumann's solution.

ACKNOWLEDGEMENT

The support of the National Science Foundation through Grant No. GK 1281 is acknowledged.

REFERENCES

1. K. A. RATHJEN, Heat conduction with melting or freezing in a corner, Ph.D. Thesis, The City University of New York, New York (1968).
2. R. D. ZERKLE and J. E. SUNDERLAND, The effect of liquid solidification in a tube laminar-flow heat transfer and pressure drop, *J. Heat Transfer* **90**, 183–190 (1968).
3. R. T. BEAUBOUEF and A. J. CHAPMAN, Freezing of fluids in forced flow, *Int. J. Heat Mass Transfer* **10**, 1581–1587 (1967).
4. P. A. LIBBY and S. CHEN, The growth of a deposited layer on a cold surface, *Int. J. Heat Mass Transfer* **8**, 395–402 (1965).

Int J Heat Mass Transfer Vol 13, pp 218–221 Pergamon Press 1970. Printed in Great Britain

TRANSIENT HEATING OF THIN PLATES

H. J. BREAUX and P. R. SCHLEGEL

Ballistic Research Laboratories, U.S. Army Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland 21005

(*Received* 12 *March* 1969 *and in revised form* 4 *August* 1969)

NOMENCLATURE

$E_1(x)$,	exponential integral function;
erf(x),	error function;
erfc(x),	complementary error function;
h,	coefficient of surface heat transfer;
$I_\nu(x)$,	modified Bessel function of the first kind;
$J_\nu(x)$,	Bessel function;
$K_\nu(x)$,	modified Bessel function of the third kind;
k,	thermal conductivity;
Q_0,	maximum flux of heat source;

$Q(y,\beta)$, source function;
$q(x)$, $= Q(\beta x,\beta)/Q_0$, non-dimensional source function;
t, $= \alpha\tau/\beta^2$, non-dimensional time;
T, $= \theta k\delta/\beta^2 Q_0$, non-dimensional temperature;
x, $= y/\beta$, non-dimensional distance from center of heat source;
y, distance from center of heat source;
α, thermal diffusivity;
β, half width of strip source, radius of disk source, parameter in Gaussian source;
δ, thickness of plate;
ε^2, $= 2h\beta^2/k\delta$, non-dimensional coefficient of surface heat transfer;
σ, integral transform variable;
τ, time;
θ, temperature.

INTRODUCTION

SEVERAL types of conduction heat transfer applications involve the determination of transient temperature in a thin plate heated over a portion of its surface. Typical applications include the heating of metal plates with a laser or incendiary, drilling holes in metal, acetylene burning, arc welding, and friction heating of machine parts. Several solutions for disk, strip and point sources can be found in the literature [1, 2], usually in the form of a definite integral which must be solved by numerical quadrature. The two general solutions derived in this paper encompass most of the special case solutions encountered in the literature. In addition, these general solutions extend the range of applicable problems to two large classes of source functions. Some new solutions in "closed" form are presented. In addition an integral solution, obtained by P. H. Thomas [1] for strip heating, has been reduced to a closed form.

The two general problems examined include thin plates heated by sources having symmetry about the line $x = 0$ and thin plates heated by sources having circular symmetry about the origin. The source terms are required to have the following properties:

(i) The source term $q(x)$, having symmetry about the line $x = 0$, is assumed to be such that its Fourier cosine transform exists and that $\partial q/\partial x = 0$ at $x = 0$.

(ii) The source term $q(x)$, having circular symmetry about the point $x = 0$, is assumed to be such that its zero-order Hankel transform exists and that $q(0)$ is finite.

In all cases the heat loss at the surface of the plate is assumed to be subject to Newton's Law of Cooling.

SOURCES SYMMETRICAL ABOUT A LINE

The partial differential equation and boundary conditions governing this case are

$$\frac{\partial T}{\partial t} + \varepsilon^2 T = \frac{\partial^2 T}{\partial x^2} + q(x), x > 0, t > 0, \qquad (1)$$

$$T = 0, \quad x \geqslant 0, t = 0, \qquad (2)$$

$$\frac{\partial T}{\partial x} = 0, \quad x = 0, t > 0, \qquad (3)$$

$$T = \frac{\partial T}{\partial x} = 0, \quad x \to \infty, t \geqslant 0. \qquad (4)$$

By application of the Fourier cosine transform to (1) we obtain the subsidiary equation

$$\frac{\mathrm{d}\bar{T}(\sigma,t)}{\mathrm{d}t} + (\varepsilon^2 + \sigma^2)\bar{T}(\sigma,t) = \bar{q}(\sigma) \qquad (5)$$

where $\bar{T}(\sigma,t)$ is the Fourier cosine transform of $T(x,t)$ and $\bar{q}(\sigma)$ is the Fourier cosine transform of the source term. After solving (5) subject to the condition $\bar{T}(\sigma,0) = 0$, we obtain

$$T(x,t) = \sqrt{(2/\pi)}\int_0^\infty \frac{\bar{q}_1(\sigma)}{\varepsilon^2 + \sigma^2}[1 - \exp\{-t(\sigma^2 + \varepsilon^2)\}] \times \cos(x\sigma)\mathrm{d}\sigma. \qquad (6)$$

This solution is applicable for all cases having source terms as described in the introduction.

We shall consider two specific source functions having symmetry about $x = 0$.

1. A "uniform strip source" is defined by

$$q(x) = \begin{cases} 1, |x| \leqslant 1, \\ 0, |x| > 1. \end{cases} \qquad (7)$$

This source has a Fourier cosine transform given by

$$\bar{q}(\sigma) = \sqrt{(2/\pi)}\sigma^{-1}\sin\sigma \qquad (8)$$

which, upon insertion into (6), gives the integral solution obtained by Thomas. Employing the identity

$$\sin\sigma\cos x\sigma = \tfrac{1}{2}[\sin(1 + x)\sigma + \sin(1 - x)\sigma],$$

partial fractions, and the Fourier sine transform pairs (4) p. 19, (20) p. 65, (21) p. 73 and (26) p. 74 listed in Vol. 1 of [3] we obtain

$$T(x,t) = T_1(x) - T_2(x,t), \qquad (9)$$

where

$$T_1(x) = \begin{cases} \varepsilon^{-2}[1 - \mathrm{e}^{-\varepsilon}\cosh(\varepsilon x)], & 0 \leqslant x \leqslant 1, \\ \varepsilon^{-2}\mathrm{e}^{-\varepsilon x}\sinh(\varepsilon), x > 1, \end{cases} \qquad (10)$$

and

$$T_2(x,t) = \frac{1}{2\varepsilon^2}\left\{\mathrm{e}^{-t\varepsilon^2}\left[\mathrm{erf}\left(\frac{1+x}{2\sqrt{t}}\right) + \mathrm{erf}\left(\frac{1-x}{2\sqrt{t}}\right)\right]\right.$$
$$+ \frac{1}{2}\left[\mathrm{e}^{\varepsilon(1+x)}\mathrm{erfc}\left(\varepsilon t + \frac{1+x}{2\sqrt{t}}\right) - \mathrm{e}^{-\varepsilon(1+x)}\right.$$
$$\left(\varepsilon t - \frac{1+x}{2\sqrt{t}}\right) + \mathrm{e}^{\varepsilon(1-x)}\mathrm{erfc}\left(\varepsilon t + \frac{1-x}{2\sqrt{t}}\right)$$
$$\left.\left. - \mathrm{e}^{-\varepsilon(1-x)}\mathrm{erfc}\left(\varepsilon t - \frac{1-x}{2\sqrt{t}}\right)\right]\right\} \qquad (11)$$

The steady state solution is given by $T_1(x)$, which for $x = 0$, yields the special case given by Thomas. The behaviour of this solution for $x = 0$ is shown in Fig. 1.

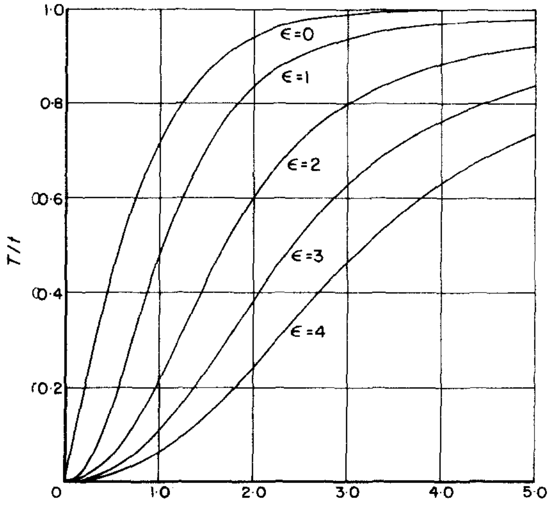

FIG. 1. Effect of stripwidth, heating time and heat transfer coefficient on maximum temperature of thin plate heated by uniform strip source.

2. We define a "Gaussian strip source" by the equation

$$q(x) = e^{-x^2}. \tag{12}$$

This source has a Fourier cosine transform given by

$$\bar{q}_1(\sigma) = \frac{1}{\sqrt{2}}\,e^{-\sigma^2/4}. \tag{13}$$

For this source we obtain the solution

$$T(x,t) = \frac{\sqrt{\pi}\exp(\varepsilon^2/4)}{4\varepsilon}\left[\!\left[e^{-\varepsilon x}\left\{\operatorname{erfc}\left(\frac{\varepsilon}{2} - x\right) - \operatorname{erfc}\left[\frac{\varepsilon}{2}\sqrt{(4t+1)} - \frac{x}{\sqrt{(4t+1)}}\right]\right\} + e^{\varepsilon x}\left\{\operatorname{erfc}\left(\frac{\varepsilon}{2} + x\right) - \operatorname{erfc}\left[\frac{\varepsilon}{2}\sqrt{(4t+1)} + \frac{x}{\sqrt{(4t+1)}}\right]\right\}\right]\!\right] \tag{14}$$

CIRCULARLY SYMMETRIC SOURCES

This case is governed by the equations

$$\frac{\partial T}{\partial t} + \varepsilon^2 T = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right) + q(x), x > 0, t > 0, \tag{15}$$

$$T = 0 \qquad x \geqslant 0, t = 0, \tag{16}$$

$$T = \frac{\partial T}{\partial x} = 0, \qquad x \to \infty, t \geqslant 0. \tag{17}$$

Applying the zero order Hankel transform to (15) and proceeding as before, we obtain

$$T(x,t) = \int_0^\infty \bar{q}(\sigma)\frac{\sigma J_0(\sigma x)}{\sigma^2 + \varepsilon^2}[1 - \exp\{-t(\sigma^2 + \varepsilon^2)\}]\,d\sigma, \tag{18}$$

where $\bar{q}(\sigma)$ is the zero-order Hankel transform of the source term $q(x)$.

Again we consider two specific sources:

1. For a "Gaussian circular source" given by (12) having a zero-order Hankel transform

$$\bar{q}(\sigma) = \frac{1}{2}e^{-\sigma^2/4} \tag{19}$$

and for the case of no cooling, ($\varepsilon = 0$), we obtain the solution

$$T(x,t) = \begin{cases} \frac{1}{4}\left\{E_1\left(\frac{x^2}{4t+1}\right) - E_1(x^2)\right\}, x > 0 \\ \frac{1}{4}\ln(4t+1), x = 0. \end{cases} \tag{20}$$

2. A "uniform disk source" is defined by

$$q(x) = \begin{cases} 1, 0 \leqslant x \leqslant 1, \\ 0, x > 1, \end{cases}$$

and its zero-order Hankel transform is

$$\bar{q}(\sigma) = J_1(\sigma)/\sigma. \tag{21}$$

Hence the solution for a disk source is given by

$$T(x,t) = \int_0^\infty \frac{J_0(\sigma x)J_1(\sigma)}{\sigma^2 + \varepsilon^2}[1 - \exp\{-t(\sigma^2 + \varepsilon^2)\}]\,d\sigma, \tag{22}$$

an integral solution given by Thomas [1]. This integral cannot be reduced further; however, for steady state, we obtain

$$T(x) = \begin{cases} \varepsilon^{-2}[1 - \varepsilon K_1(\varepsilon)I_0(\varepsilon x)], x \leqslant 1 \\ \varepsilon^{-1}I_1(\varepsilon)K_0(\varepsilon x), \qquad x > 1 \end{cases} \tag{23}$$

The general case, (22), can be approximated by a definite integral over finite limits, i.e.

$$T(x,t) = \int_0^U \frac{J_0(\sigma x)J_1(\sigma)}{\sigma^2 + \varepsilon^2} \times [1 - \exp\{-t(\sigma^2 + \varepsilon^2)\}]\,d\sigma + E_1. \tag{24}$$

It can be shown that if U is chosen by the formula

$$U = \begin{cases} \varepsilon\left[\exp\left\{\dfrac{E\varepsilon^2\sqrt{x}}{A}\right\} - 1\right]^{-\frac{1}{2}}, & x > 0 \\ \left[\dfrac{3E}{2\sqrt{(2)}\pi A}\right]^{-\frac{2}{3}}, & x = 0, \end{cases} \tag{25}$$

where

$$A = 2^{5/4}(1 + e^{-1})/\pi^2.$$

and E is an acceptable error bound, then

$$|E_1| \leqslant E.$$

A comparison of the temperature rise at $x = 0$ for no cooling, ($\varepsilon = 0$), for the four source functions is shown in Fig. 2.

REFERENCES

1. P. H. THOMAS, Some conduction problems in the heating of small areas on large solids, *Q. Jl Mech. Appl. Math.* **10**, 482–493 (1957).
2. H. S. CARSLAW and J. C. JAEGER, *Conduction of Heat in Solids.* Oxford University Press, London (1959).
3. A. ERDELYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI, *Tables of Integral Transforms*, Vol. 1 and Vol. 2. McGraw-Hill, New York (1954).

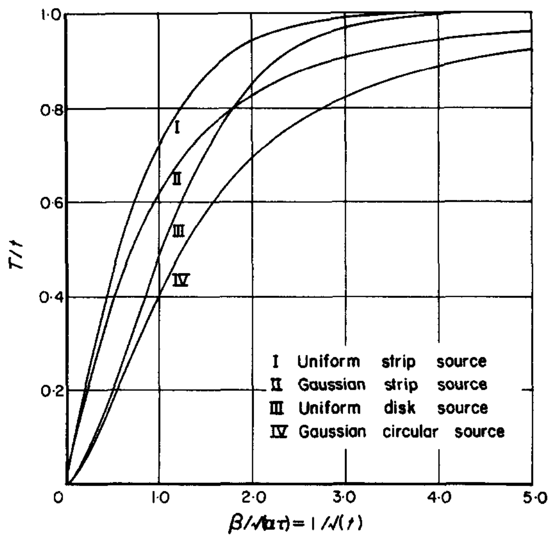

FIG. 2. Effect of source size, source type and heating time on maximum temperature of thin plate with no cooling ($\varepsilon = 0$).